THE WAVE EQUATION AND THE GREEN'S DYADIC FOR BOUNDED MAGNETOPLASMAS

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ABSTRACT

In studies of electromagnetic wave propagation and radiation in magnetoplasmas, the wave equation takes the form of a dyadic-vector Helmholtz equation. The investigation here shows that the dyadic-vector Helmholtz equation is solvable by the separation method in four cylindrical coordinate systems. Solutions in the form of complete sets of eigenfunctions are possible when boundary surfaces are present. For problems involving current sources in the plasma, the Green's dyadics for finite or semifinite domains can be constructed from the complete sets of eigenfunctions which are solutions to the homogeneous equation. The Green's dyadic for infinite domain is also shown to be obtainable from that for a semifinite domain through a limiting process.

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INTRODUCTION

The presence of a static magnetic field in a plasma region results in an effective electric conductivity which is of dyadic form. Assuming monochomatic waves, the equation describing the waves, generated by a source, $\overline{J}_s$, in such an anisotropic medium may be written as

$$\nabla \times \nabla \times \overline{E} - \overline{\overline{k}} \cdot \overline{E} = \overline{J}_s . \tag{1}$$

Written in matrix form, the dyadic $\overline{\overline{k}}$ is

$$\overline{\overline{k}} = \begin{pmatrix} k_\perp & k_T & 0 \\ -k_T & k_\perp & 0 \\ 0 & 0 & k_{\parallel} \end{pmatrix} . \tag{2}$$

Assuming spatial homogeneity, the parameters $k_\perp$, k_T, and $k_{\parallel}$ are constants with respect to time and space coordinates.

Solutions for Eq. (1) in terms of auxiliary Green's function for infinite domain have been discussed by Bunkin(1) and subsequently extended by Chow(2) and Brandstater.(3) However, the solutions of Eq. (1) for a bounded region have proved to be more difficult to obtain. The studies dealt with here reveal that, in order to solve for a finite domain or semifinite domain Green's function, a better understanding of the free wave equation, $\overline{J}_s = 0$ in Eq. (1), is needed, and that the Green's function may be constructed from the solutions of the homogeneous equation.

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(1) F. V. Bunkin, Soviet Phys., JETP, 5, 277, (1957).

(2) Y. Chow, Trans. I.R.E., AP-10, 464, (1962).

(3) J. J. Brandstater, An Introduction to Waves, Rays, and Radiation in Plasma, (McGraw-Hill Book Company, Inc., New York, 1963) Chapter 9.

THE HOMOGENEOUS EQUATION

The homogeneous equation describing free wave propagation is

$$\nabla \times \nabla \times \overline{E} - \overline{\overline{k}} \cdot \overline{E} = 0. \qquad (3)$$

Eq. (3) is seen to resemble a vector Helmholtz equation except that $\overline{\overline{k}}$ is a dyadic. It is well known that the scalar Helmholtz equation is separable in eleven coordinate systems, and that the vector Helmholtz equation is separable in only six coordinate systems.[4] Despite the fact that the dyadic-vector Helmholtz equation has been frequently encountered in connection with the studies of crystal materials and plasma fields, and that its solutions have been obtained and used extensively for problems involving boundaries in the rectangular coordinate systems and the circular cylindrical coordinate systems,[5,6,7] additional investigation into the separability of the dyadic-vector Helmholtz equation is desirable. The separability of Eq. (3) will be studied here, since by determining the coordinate systems in which this equation is separable one not only gains the knowledge of exactly in what coordinate systems the equation is solvable by a separation method, but one also hopefully attempts solutions in the form of eigenfunctions when boundaries are involved. The eigenfunction solutions will be of great help in constructing the finite domain or semifinite domain Green's dyadics.

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(4) P.M. Morse, and H. Feshbach, Methods of Theoretical Physics, (McGraw-Hill Book Company, Inc., New York, 1953) Vol. II, Chapter 13.

(5) W.P. Allis, S.J. Buchsman, and A. Bers, Waves in Anisotropic Plasmas, (M.I.T. Press, Cambridge, Massachusetts, 1963) Part . II.

(6) H. Suhl, and L. R. Walker, Bell System Tech. J., 33, 579-939-1133, (1954).

(7) A.A. Th.M. van Trier, Appl. Sc. Research (Netherland), B3, 305, (1953).

In the application of boundary value problems, separation into the form that conveniences the fitting of boundary surfaces is most desirable. Hence, it is advisable to separate this dyadic-vector Helmholtz equation in terms of longitudinal, $\overline{L}$, and transverse, $\overline{M}$ and $\overline{N}$, vector components.

The first term in Eq. (3) is a vector operating term, $\nabla \times \nabla \times \overline{E}$. A review of the separability of a vector Helmholtz equation shows that the coordinate system in which this vector operating term facilitates separation must be a coordinate system in which one of the scale factors is unity, and that the ratio of the remaining two scale factors must be independent of the coordinate corresponding to the unity scale factor. The six coordinate systems which meet these requirements are the spherical, the conical, and the four cylindrical coordinate systems.

Pertaining to magnetoactive plasma, Eq. (2) implies that the static magnetic field is in the direction parallel or anti-parallel to the coordinate corresponding to the unity scale factor. Without losing generality, this coordinate is denoted ξ_3 , and its unit vector, $\overline{a}_3$. A close examination shows that only four out of the six coordinate systems are physically realizable for such alignment of the static magnetic field; namely, the four cylindrical coordinate systems including the rectangular, the circular cylindrical, the elliptical cylindrical, and the parabolic cylindrical coordinate systems. In each system, ξ_3 corresponds to the z axis.

It may first seem to be pessimistic that the number of permissible coordinate systems has been reduced to only four from eleven right at the onset. Fortunately, however, it turns out that no other restriction will be imposed that further reduces the number of permissible coordinate systems.

In attempting the solution of Eq. (3), the difficulty lies in the fact that each term in the equation is a purely transverse vector, while due to the dyadic $\bar{\bar{k}}$, the vector field, $\bar{E}$, in general, is not entirely transverse. Since it is desirable to separate the equation in terms of transverse and longitudinal components, $\bar{E}$ must be expressed in terms of all three vector components $\bar{L}$, $\bar{M}$ and $\bar{N}$, i.e.:

$$
\begin{aligned}
\bar{E} &= \bar{L} + \bar{M} + \bar{N} , \\
\bar{L} &= -\nabla_{\perp}\phi - \nabla_{\parallel}\phi , \\
\bar{M} &= \nabla_{\perp}\psi \times \bar{a}_3 , \\
\bar{N} &= \nabla_{\perp}(\nabla_{\parallel}\cdot\bar{a}_3\chi) - (\nabla_{\perp}^2\chi)\bar{a}_3 ;
\end{aligned}
\tag{4}
$$

where ϕ , ψ , and χ are three scalar functions to be determined. The subscript $\perp$ indicates the components of operator or vector which are perpendicular to $\bar{a}_3$, whereas $\parallel$ indicates parallel to $\bar{a}_3$.

Expanding $\bar{\bar{k}}\cdot\bar{E}$ into vector form, Eq. (3) may be broken into two equations, one contains the $\perp$ -vectors, the other contains the $\parallel$ -vectors. It is also recognized that Eq. (3) implies:

$$\nabla\cdot\left(\bar{\bar{k}}\cdot\bar{E}\right) = 0 , \tag{5}$$

which yields a third equation. After some manipulation, the three basic equations become.

$$\nabla_{\perp}^2(\nabla_{\perp}^2\psi) + \nabla_{\parallel}^2(\nabla_{\perp}^2\psi) + k_T\nabla_{\parallel}\cdot\bar{a}_3(\nabla_{\perp}^2\chi) - k_T\nabla_{\perp}^2\phi + k_{\perp}(\nabla_{\perp}^2\psi) = 0, \tag{6}$$

$$\nabla_{\perp}^2(\nabla_{\perp}^2\chi) + \nabla_{\parallel}^2(\nabla_{\perp}^2\chi) + k_{\parallel}\nabla_{\perp}^2\chi + k_{\parallel}\nabla_{\parallel}\cdot\bar{a}_3\phi = 0, \tag{7}$$

$$k_T(\nabla_{\perp}^2\psi) + (k_{\parallel} - k_{\perp})\nabla_{\parallel}\cdot\bar{a}_3(\nabla_{\perp}^2\chi) + k_{\perp}\nabla_{\perp}^2\phi + k_{\parallel}\nabla_{\parallel}^2\phi = 0. \tag{8}$$

Close examination of Eqs. (6) to (8) shows that solutions may be obtained if the three scalar functions each satisfies:

$$\nabla^2(\Phi,\Psi,\chi) + T_m^2(\Phi,\Psi,\chi) = 0\ , \tag{9}$$

or

$$\nabla_\perp^2(\Phi,\Psi,\chi) + K_m^2(\Phi,\Psi,\chi) = 0 \tag{10a}$$

and

$$\nabla_\parallel^2(\Phi,\Psi,\chi) + k_m^2(\Phi,\Psi,\chi) = 0\ , \tag{10b}$$

with

$$K_m^2 = T_m^2 - k_m^2\ , \tag{10c}$$

where k_m^2 is the separation constant for separation of ξ_3. T_m^2 in Eqs. (9) or (10c) must satisfy an eighth order determinant equation

$$\begin{vmatrix} (T_m^2 - k_\perp) & -k_T & k_T \\ 0 & (T_m^2 - k_m^2)(T_m^2 - k_\parallel) & -k_\parallel k_m^2 \\ k_T(T_m^2 - k_m^2) & (k_\parallel - k_\perp)(T_m^2 - k_m^2) & k_\perp(T_m^2 - k_m^2) + k_\parallel k_m^2 \end{vmatrix} = 0. \tag{11}$$

The order of Eq. (11) appears to be too high to be readily solved at first, but it turns out that the resulting secular equation is only of fourth order in T_m, since the other four roots, $T_m^2 = k_m^2$, and $T_m^2 = 0$, are trivial and may be discarded. The secular equation yield by Eq. (11) is

$$(T_m^2 - k_m^2 - k_\parallel)\left[k_T^2 - k_\perp(T_m^2 - k_\perp)\right] - k_\parallel k_m^2 (T_m^2 - k_\perp) = 0. \tag{12}$$

Eq. (12) may be readily solved for T_m^2 in terms of k_m^2, or for k_m^2 in terms of T_m^2 depending upon the manner of the boundaries set up in the problem. Let the solutions of the scalar functions be:

$$\psi = A\,\psi_\perp(\xi_1, \xi_2, \kappa_m)\,\psi_\parallel(\xi_3, k_m), \tag{13}$$

$$\chi = B\,\chi_\perp(\xi_1, \xi_2, \kappa_m)\,\chi_\parallel(\xi_3, k_m), \tag{14}$$

$$\phi = C\,\phi_\perp(\xi_1, \xi_2, \kappa_m)\,\phi_\parallel(\xi_3, k_m). \tag{15}$$

If the boundaries are parallel to the ξ_3 = constant surfaces, $\psi_\parallel$, $\chi_\parallel$, and $\phi_\parallel$ are sets of eigenfunctions and k_m are the eigenvalues with index m; then κ_m, obtained from Eqs. (12) and (10c), will describe the dispersion relation for propagation in the (ξ_1, ξ_2) space. Conversely, if the boundary surfaces are perpendicular to ξ_3 = constant surfaces, $\psi_\perp$, $\chi_\perp$, and $\phi_\perp$ will consist of sets of eigenfunctions with κ_m consisting of the eigenvalues. $\psi_\parallel$, $\chi_\parallel$, and $\phi_\parallel$ describe the propagation in $\bar{a}_3$ direction with k_m being the parameter describing the dispersion relation. In either case there will be another eigenvalue with index n, resulting from the separation of Eq. (10b), which is not apparant in Eqs. (13) to (15). Of course, when the boundary is a completely self-enclosed one, there are three sets of eigenvalues with indices m, n, and ℓ. The solutions ψ, χ, and ϕ are not entirely independent. By substitution of Eqs. (13) to (15) into Eqs. (6) to (8), it is possible to obtain a functional relation between the constants A, B, and C, thus reducing the number of arbitrary constants to one.

Without restricting the generality of the two succeeding sections on the inhomogeneous equation and the Green's function, and on the infinite domain Green's dyadic, a readily understandable illustration is that of a plasma region bounded by two parallel, infinitely large, conducting plates of finite separation, d, with a static magnetic field imposed upon the plasma in the direction normal to the boundary plates. The solution for outgoing waves can be found in a circular cylindrical coordinate system. Assuming the origin of the coordinate system is located midway between the plates

$$\psi = \sum_{m,n} A_{mn} H_n^{(2)}(\kappa_m r) e^{jn\theta} \cos k_m z \,, \tag{16}$$

$$\chi = \sum_{m,n} B_{mn} H_n^{(2)}(\kappa_m r) e^{jn\theta} \sin k_m z \,, \tag{17}$$

$$\phi = \sum_{m,n} C_{mn} H_n^{(2)}(\kappa_m r) e^{jn\theta} \cos k_m z \,; \tag{18}$$

with two sets of eigenvalues, i. e.

$$\begin{aligned} n &= 0, \pm 1, \pm 2, \text{----} \,, \\ k_m &= \frac{m\pi}{d}\,; \quad m = 0, 1, 2, \text{----} \,. \end{aligned} \tag{19}$$

As stated above, functional relations between A_{mn}, B_{mn}, and C_{mn} may be obtained by substituting Eqs. (16) to (18) into Eqs. (6) to (8). Since $\cos k_m z$ or $\sin k_m z$ when summed on m constitutes a complete set of eigen-functions, this complete set is a complete solution of Eq. (10b). Also, the functions $e^{jn\theta}$ when summed on n yield a complete set of eigen-functions that satisfies an equation resulting from separation of θ from Eq. (10a). Therefore, by virtue of the completeness theorem for several variables,[8] the functions ψ, χ, and ϕ as shown in Eqs. (16) to (18) are complete sets which satisfy Eq. (9). Consequently, the wave field $\bar{E}$, obtained from Eq. (4), having three orthogonal components, namely $\nabla_\perp \psi \times \bar{a}_3$, $\nabla_\perp(\nabla_\parallel \cdot \bar{a}_3 \chi - \phi)$, and $\bar{a}_3(\nabla_\perp^2 \chi + \nabla_\parallel \cdot \bar{a}_3 \phi)$, with each component consisting of complete sets, must by itself be complete. Hence, $\bar{E}$ so obtained is a complete solution to Eq. (3). In addition, it can also be shown

(8) See for example, R. Courant, and D. Hilbert, Methods of Mathematical Physics, (Interscience Publishers, Inc., New York, 1953), Vol. I, p. 56.

that for a physically realizable problem, the boundary conditions require either the tangential component of electric field or the tangential component of the magnetic field vanishes at the boundary; from this, the solution obtained by Eq. (4) can be proved to be unique.[9]

INHOMOGENEOUS EQUATION AND THE GREEN'S FUNCTION

When a source $\overline{\overline{J}}_s(\bar{r})$ is presented in the bounded region, it can be shown that Eq. (1) is solvable in terms of an integral representation:

$$\overline{E}(\bar{r}) = \int_{V_0} \overline{\overline{G}}(\bar{r}|\bar{r}_0) \cdot \overline{\overline{J}}_s(\bar{r}_0)\, dv_0 \, , \tag{20}$$

where the kernel $\overline{\overline{G}}(\bar{r}|\bar{r}_0)$ is the usual Green's dyadic function except that instead of satisfying Eq. (1) with a dyadic impulse source, it satisfies the following:

$$\nabla \times \nabla \times \overline{\overline{G}} - \overline{\overline{k}}_c \cdot \overline{\overline{G}} = \overline{\overline{\mathcal{J}}}\, \delta(\bar{r} - \bar{r}_0) \, , \tag{21}$$

where $\overline{\overline{\mathcal{J}}}$ is the idemfactor and $\overline{\overline{k}}_c$ is the conjugate of $\overline{\overline{k}}$. The use of the conjugate of $\overline{\overline{k}}$ in Eq. (21) is necessary if it is desired to include the cases where $\overline{\overline{k}}$ is not Hermitian, (see Appendix). In addition to satisfying Eq. (21), the Green's function must also satisfy the same boundary condition that the field satisfies.

The derivation of a Green's function to be discussed here depends upon whether there are boundary surfaces parallel to the $\xi_3 = Const$ surface. For

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(9) See for example, J. A. Stratton, Electromagnetic Theory, (McGraw-Hill Book Company, Inc., New York, 1941) pp. 486-488.

brevity, only the case with boundary surface parallel to the $\xi_3 = Const.$ surface will be derived here. It is assured that the Green's dyadic for the case of no boundary surface parallel to $\xi_3 = Const.$ surface may also be derived with the same technique, except for some minor modifications.

In view of the form of the solutions to the homogeneous equation, and in view of the fact that the three scalar functions are not independent functions, it is proposed that the Green's dyadic takes the form:

$$\bar{\bar{G}} = \bar{\bar{G}}_M + \bar{\bar{G}}_N + \bar{\bar{G}}_L, \tag{22}$$

where

$$\bar{\bar{G}}_M = \sum_{m,n} \left\{ (\nabla_\perp \varphi_{mn} \times \bar{a}_3) f_m \right\} \bar{F}_{mn}(\xi_1^\circ, \xi_2^\circ, \xi_3^\circ), \tag{23}$$

$$\bar{\bar{G}}_N = \sum_{m,n} \left\{ (\nabla_\perp \varphi_{mn})(\nabla_\parallel \cdot \bar{a}_3 g_m) - (\nabla_\perp^2 \varphi_{mn}) g_m \bar{a}_3 \right\} \bar{G}_{mn}(\xi_1^\circ, \xi_2^\circ, \xi_3^\circ), \tag{24}$$

$$\bar{\bar{G}}_L = \sum_{m,n} \left\{ \nabla_\perp \varphi_{mn} f_m + \varphi_{mn} (\nabla_\parallel f_m) \right\} \bar{H}_{mn}(\xi_1^\circ, \xi_2^\circ, \xi_3^\circ) \quad \text{, and} \tag{25}$$

where $\bar{F}_{mn}$, $\bar{G}_{mn}$, and $\bar{H}_{mn}$ are functions of source coordinates only.

φ_{mn} is a two-variable function of variables ξ_1 and ξ_2, satisfying Eq. (10a). The ξ_3 dependent functions f_m and g_m are the two independent solutions of Eq. (10b); their relation is dictated by whether there is a closed or open boundary in ξ_3. For the case of closed boundary in ξ_3, the relation is

$$\nabla_\parallel \cdot \bar{a}_3 g_m = \mp k_m f_m \quad , \quad \nabla_\parallel \cdot \bar{a}_3 f_m = \pm k_m g_m. \tag{26}$$

The choice of upper or lower sign in Eq. (26) depends on the type of boundary condition and the choice of coordinate origin in the problem. To be general, both signs will be kept throughout this derivation. Finally, the index n in Eqs. (23) to (25) may be a single index or a double index, depending upon whether the boundary perpendicular to ξ_3 is open or closed.

The vectors $\overline{M}$, $\overline{N}$, and $\overline{L}$ are not necessarily orthogonal in space, but $\nabla_\perp \varphi_{mn} \times \overline{a}_3$, $\nabla_\perp \varphi_{mn}$, and $\overline{a}_3$ are three orthogonal vectors. If a unit vector $\overline{b}$, and a two-variable-dependent function $\rho_{mn}(\xi_1, \xi_2)$ are defined such that

$$\nabla_\perp \varphi_{mn}(\xi_1,\xi_2) = \rho_{mn}(\xi_1,\xi_2)\,\overline{b} \tag{27}$$

then the unit vectors $\overline{b}$, $\overline{b} \times \overline{a}_3$, and $\overline{a}_3$ are mutually orthogonal in space. Multiplication of these unit vectors $\overline{b}$, $\overline{b} \times \overline{a}_3$, and $\overline{a}_3$ in turn into Eq. (20) yields a set of three mutually orthogonal equations;

$$\sum_{m,n}\Big\{-(\nabla^2+k_\perp)(\rho_{mn}f_m)\overline{F}_{mn} - k_T(\rho_{mn}f_m)(k_m\overline{G}_{mn}) + k_T(\rho_{mn}f_m)\overline{H}_{mn} = \overline{b}\times\overline{a}_3\,\delta(\overline{r}-\overline{r}_0), \tag{28}$$

$$\sum_{m,n}\Big\{k_T(\rho_{mn}f_m)\overline{F}_{mn} - (\nabla^2+k_\perp)(\rho_{mn}f_m)(k_m\overline{G}_{mn}) + k_\perp(\rho_{mn}f_m)\overline{H}_{mn}\Big\} = \overline{b}\,\delta(\overline{r}-\overline{r}_0), \tag{29}$$

$$\sum_{m,n}\Big\{(\nabla^2+k_\parallel)(\nabla^2+k_m^2)(\varphi_{mn}g_m)(k_m\overline{G}_{mn}) - k_\parallel k_m^2(\varphi_{mn}g_m)\overline{H}_{mn}\Big\} = k_m\overline{a}_3\,\delta(\overline{r}-\overline{r}_0). \tag{30}$$

In Eqs. (28) to (30), the relation given by Eqs. (22) to (26) has been substituted. The operator ∇^2 is a three dimensional operator operating on the observer coordinate functions only, i.e., $(\rho_{mn}f_m)$ or $(\varphi_{mn}g_m)$. When operation on the source coordinate is needed, the operators will be distinguished by a superscript o.

In order to express the ξ_3^o dependent functions explicitly, and to express the source coordinate functions in component form, $\overline{F}_{mn}$, $\overline{G}_{mn}$, and $\overline{H}_{mn}$ may be assumed:

$$\overline{F}_{mn}=F_{mn}^{x}(\xi_1^o,\xi_2^o)f_m^{x}(\xi_3^o)\,\bar{b}\times\bar{a}_3+F_{mn}^{\perp}(\xi_1^o,\xi_2^o)f_m^{\perp}(\xi_3^o)\,\bar{b}+F_{mn}^{\parallel}(\xi_1^o,\xi_2^o)f_m^{\parallel}(\xi_3^o)\,\bar{a}_3\,, \tag{31a}$$

$$\overline{G}_{mn}=G_{mn}^{x}(\xi_1^o,\xi_2^o)g_m^{x}(\xi_3^o)\,\bar{b}\times\bar{a}_3+G_{mn}^{\perp}(\xi_1^o,\xi_2^o)g_m^{\perp}(\xi_3^o)\,\bar{b}+G_{mn}^{\parallel}(\xi_1^o,\xi_2^o)g_m^{\parallel}(\xi_3^o)\,\bar{a}_3\,, \tag{31b}$$

$$\overline{H}_{mn}=H_{mn}^{x}(\xi_1^o,\xi_2^o)h_m^{x}(\xi_3^o)\,\bar{b}\times\bar{a}_3+H_{mn}^{\perp}(\xi_1,\xi_2)h_m^{\perp}(\xi_3^o)\,\bar{b}+H_{mn}^{\parallel}(\xi_1,\xi_2)h_m^{\parallel}(\xi_3^o)\,\bar{a}_3\,. \tag{31c}$$

After some vector manipulations, Eqs. (28) to (30) are broken into a set of nine sixth order equations:

$$\sum_{m,n}\mathcal{M}(\rho_{mn}f_m)^2(\varphi_{mn}g_m)F_{mn}^{x}f_m^{x}=\sum_{m,n}\left\{k_\parallel k_m^2\nabla^2-k_\perp\nabla^2(\nabla^2+k_\parallel+k_m^2)\right\}(\rho_{mn}f_m)(\varphi_{mn}g_m)\delta(\bar{r}-\bar{r}_o)\,, \tag{32}$$

$$\sum_{m,n}\mathcal{M}(\rho_{mn}f_m)^2(\varphi_{mn}g_m)F_{mn}^{\perp}f_m^{\perp}=\sum_{m,n}\left\{-k_T\nabla^2(\nabla^2+k_\parallel+k_m^2)\right\}(\rho_{mn}f_m)(\varphi_{mn}g_m)\delta(\bar{r}-\bar{r}_o)\,, \tag{33}$$

$$\sum_{m,n}\mathcal{M}(\rho_{mn}f_m)^2(\varphi_{mn}g_m)F_{mn}^{\parallel}f_m^{\parallel}=\sum_{m,n}\pm k_T k_m\nabla^2(\rho_{mn}f_m)^2\,\delta(\bar{r}-\bar{r}_o)\,, \tag{34}$$

$$\sum_{m,n}\mathcal{M}(\rho_{mn}f_m)(\varphi_{mn}g_m)G_{mn}^{x}g_m^{x}=\sum_{m,n}\pm k_T k_\parallel k_m(\varphi_{mn}g_m)\,\delta(\bar{r}-\bar{r}_o)\,, \tag{35}$$

$$\sum_{m,n}\mathcal{M}(\rho_{mn}f_m)(\varphi_{mn}g_m)G_{mn}^{\perp}g_m^{\perp}=\sum_{m,n}\mp k_\parallel k_m(\nabla^2+k_\perp)(\varphi_{mn}g_m)\,\delta(\bar{r}-\bar{r}_o)\,, \tag{36}$$

$$\sum_{m,n}\mathcal{M}(\rho_{mn}f_m)(\varphi_{mn}g_m)G_{mn}^{\parallel}g_m^{\parallel}=\sum_{m,n}\left\{k_T^2+k_\perp(\nabla^2+k_\perp)\right\}(\rho_{mn}f_m)\,\delta(\bar{r}-\bar{r}_o)\,, \tag{37}$$

$$\sum_{m,n}\mathcal{M}(\rho_{mn}f_m)(\varphi_{mn}g_m)H_{mn}^{x}h_m^{x}=\sum_{m,n}k_T(\nabla^2+k_\parallel)(\nabla^2+k_m^2)(\varphi_{mn}g_m)\,\delta(\bar{r}-\bar{r}_o)\,, \tag{38}$$

$$\sum_{m,n}\mathcal{M}(\rho_{mn}f_m)(\varphi_{mn}g_m)H_{mn}^{\perp}h_m^{\perp}=\sum_{m,n}-(\nabla^2+k_\parallel)(\nabla^2+k_\perp)(\nabla^2+k_m^2)(\varphi_{mn}g_m)\,\delta(\bar{r}-\bar{r}_o)\,, \tag{39}$$

$$\sum_{m,n}\mathcal{M}(\rho_{mn}f_m)(\varphi_{mn}g_m)H_{mn}^{\parallel}h_m^{\parallel}=\sum_{m,n}\pm k_m\left\{k_T^2+(\nabla^2+k_\perp)^2\right\}(\rho_{mn}f_m)\,\delta(\bar{r}-\bar{r}_o)\,, \tag{40}$$

where the operator $\mathcal{M}$ is an observer coordinate operator, it can be considered to be operating on any one of the two or three observer coordinate functions immediately to its right:

$$\mathcal{M} = \begin{cases} k_\perp T_m^2 (T_m^2 - {}_2T_m^2)(\nabla^2 - T_m^2) \quad , \quad \text{if } T_m^2 \longrightarrow {}_1T_m^2 ; \\ k_\perp T_m^2 (T_m^2 - {}_1T_m^2)(\nabla^2 - T_m^2) \quad , \quad \text{if } T_m^2 \longrightarrow {}_2T_m^2 ; \end{cases} \tag{41}$$

where ${}_1T_m^2$ and ${}_2T_m^2$ are the two non-trivial roots of Eq. (12). In view of Eq. (9), all operators ∇^2 to the right of the equality sign in Eqs. (32) to (40) are replaced by $(-T_m^2)$. Substitute Eq. (41) into Eqs. (32) to (40) and drop out the functions common to both sides of the equality sign. Multiply both sides by f_m^* or g_m^* , whichever one is appropriate. Then integrate over the entire bounded ξ_3 space, utilizing the orthogonal properties of the eigenfunctions f_m and g_m :

$$\begin{aligned} &\int f_m f_m^* \, dV_{\xi_3} = \Lambda_m^2 , \\ &\int g_m g_m^* \, dV_{\xi_3} = \Lambda_m^2 , \end{aligned} \tag{42}$$

where Λ_m^2 is the normalization factor. The asterisk (*) indicates the complex conjugate. The integration yields distinct solutions for the ξ_3° dependent functions:

$$\begin{aligned} &f_m^{\parallel}(\xi_3^\circ) = \mathfrak{f}_m^{\parallel}(\xi_3^\circ) = g_m^{\parallel}(\xi_3^\circ) = \frac{1}{\Lambda_m^2} g_m^*(\xi_3^\circ) , \\ &f_m^{\times}(\xi_3^\circ) = f_m^{\perp}(\xi_3^\circ) = g_m^{\times}(\xi_3^\circ) = g_m^{\perp}(\xi_3^\circ) = \mathfrak{g}_m^{\times}(\xi_3^\circ) = \mathfrak{g}_m^{\perp}(\xi_3^\circ) = \frac{1}{\Lambda_m^2} f_m^*(\xi_3^\circ) . \end{aligned} \tag{43}$$

Eq. (42) together with Eq. (43) indicate that the functions $f_m(\xi_3) f_m^*(\xi_3^\circ)$ and $g_m(\xi_3) g_m^*(\xi_3^\circ)$ are one dimensional scalar Green's function, both satisfying a one dimensional equation:

$$\nabla_{\parallel}^2 (f_m f_m^*, g_m g_m^*) + k_m^2 (f_m f_m^*, g_m g_m^*) = \delta(\xi_3 - \xi_3^\circ) . \tag{44}$$

After integrating out the ξ_3 and ξ_3° dependent functions, Eqs. (32) to (40) become convenient two dimensional simultaneous equations involving (ξ_1, ξ_2) and $(\xi_1^\circ, \xi_2^\circ)$ only. It may be demanded that

$$\left\{\nabla_\perp^2 + K_m^2\right\}(\rho_{mn}\,\gamma_{mn}^\times)(\bar{b}\times\bar{a}_3)(\bar{b}\times\bar{a}_3) = -K_m^2(\nabla_\perp^2 + K_m^2)\,\|\varphi_{mn}\,\sigma_{mn}^\times\|\,(\bar{b}\times\bar{a}_3)(\bar{b}\times\bar{a}_3), \tag{45}$$

and

$$\left\{\nabla_\perp^2 + K_m^2\right\}(\rho_{mn}\,\gamma_{mn}^\perp)\,\bar{b}\,\bar{b} = K_m^2\,(\nabla_\perp^2 + K_m^2)\,\|\varphi_{mn}\,\sigma_{mn}^\perp\|\,\bar{b}\,\bar{b}, \tag{46}$$

where the two double bars bracketing a function indicates only the scalar is being considered. γ_{mn} and σ_{mn} represent anyone of the $(\xi_1^\circ, \xi_2^\circ)$ functions corresponding to ρ_{mn} and φ_{mn} respectively . With Eqs. (45) and (46), all nine equations in Eqs. (32) to (40) may be represented by one symbolic equation:

$$\left\{\nabla_\perp^2 + T_m^2 - k_m^2\right\} g_{mn}(\xi_1, \xi_2 | \xi_1^\circ, \xi_2^\circ) = (\text{Constant})\,\delta(\xi_1 - \xi_1^\circ)\,\delta(\xi_2 - \xi_2^\circ). \tag{47}$$

One thus has reduced the problem of solving the Green's dyadic to one of searching for an appropriate two-dimensional scalar Green's function. Exact solutions of Eq. (47) depend upon the coordinate systems employed and the type of boundary perpendicular to $\xi_3 = \text{Const.}$ surfaces considered. In general, it can be written symbolically:

$$g_{mn} = \begin{cases} \dfrac{(\text{Constant})}{\nu^2}\,\varphi_{mn\ell}(\xi_1, \xi_2)\,\tilde{\varphi}_{mn\ell}(\xi_1^\circ, \xi_2^\circ) & \text{, for closed boundary in } \perp; \\[2ex] \dfrac{(\text{Constant})}{\nu^2}\,\varphi_{mn}(\xi_1, \xi_2)\,\tilde{\varphi}_{mn}(\xi_1^\circ, \xi_2^\circ) & \text{, for open boundary in } \perp. \end{cases} \tag{48}$$

In the case of a closed boundary in $\perp$, $\tilde{\varphi}_{mn\ell} = \varphi_{mn\ell}^*$ is the complex conjugate of $\varphi_{mn\ell}$; and $\nu^2 = \Lambda_n^2 \Lambda_\ell^2$. Λ_n^2 and Λ_ℓ^2 are the two normalization factors. In the case of an open boundary, φ_{mn} and $\tilde{\varphi}_{mn}$ are the two independent solutions of Eq. (10a) and ν^2 is a constant involving the Wronskian of the two independent solutions. Let

$$\mathcal{T}_{mn}(\xi_1,\xi_2|\xi_1^\circ,\xi_2^\circ) = \begin{cases} \varphi_{mn\ell}(\xi_1,\xi_2)\,\tilde{\varphi}_{mn\ell}(\xi_1^\circ,\xi_2^\circ), & \text{for closed boundary in } \perp; \\ \varphi_{mn}(\xi_1,\xi_2)\,\tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ), & \text{for open boundary in } \perp. \end{cases} \tag{49}$$

The complete Green's dyadic is found to be

$$\begin{aligned} \bar{\bar{G}}(\bar{r}|\bar{r}_0) = \sum_{m,n,\ell} \frac{1}{\tilde{\eta}_m \nu^2 \Lambda_m^2} \Bigg\{ & \frac{K_m^2 - k_{\parallel} + \frac{k_{\parallel}}{k_{\perp}} k_m^2}{K_m^2} (\nabla_\perp \times \bar{a}_3)(\nabla_\perp^\circ \times \bar{a}_3) \\ & + \frac{T_m^2 - k_\perp}{k_\perp K_m^2} (K_m^2 - k_\parallel)\,(\nabla_\perp)(\nabla_\perp^\circ) \\ & + \frac{1}{k_\perp k_m^2}\left[k_T^2 - (k_\perp - k_m^2)(T_m^2 - k_\perp)\right](\nabla_\parallel)(\nabla_\parallel^\circ) \\ & - \frac{k_T}{k_\perp}\left(\frac{K_m^2 - k_\parallel}{K_m^2}\right)\left[(\nabla_\perp \times \bar{a}_3)(\nabla_\perp^\circ) + (\nabla_\perp)(\nabla_\perp^\circ \times \bar{a}_3)\right] \\ & - \frac{k_T}{k_\perp}\left[(\nabla_\perp \times \bar{a}_3)(\nabla_\parallel^\circ) + (\nabla_\parallel)(\nabla_\perp^\circ \times \bar{a}_3)\right] \\ & + \frac{T_m^2 - k_\perp}{k_\perp}\left[(\nabla_\perp)(\nabla_\parallel^\circ) + (\nabla_\parallel)(\nabla_\perp^\circ)\right] \Bigg\} \mathcal{T}_{mn}(\xi_1,\xi_2|\xi_1^\circ,\xi_2^\circ)\, f_m(\xi_3) f_m^*(\xi_3^\circ)\,; \end{aligned} \tag{50}$$

where

$$\tilde{\eta}_m = \begin{cases} {}_1T_m^2 - {}_2T_m^2 , & \text{if } T_m^2 \rightarrow {}_1T_m^2 ; \\ {}_2T_m^2 - {}_1T_m^2 , & \text{if } T_m^2 \rightarrow {}_2T_m^2 . \end{cases} \tag{50a}$$

Symbolically, Eq. (50) may be written as

$$\bar{\bar{G}} = \sum_m \sum_n \sum_\ell \sum_i \sum_j \left\{ S^{ij}_{(k_m^2, K_m^2)} (\bar{\mathcal{L}}^i)(\bar{\mathcal{L}}_0^j)\, \mathcal{T}_{mn} f_m f_m^* \right\}, \tag{51}$$

where $\bar{\mathcal{L}}^i$ are the space coordinate differential operators, i.e. $\nabla_\perp \times \bar{a}_3$,

$\dot{\nabla}_{\parallel}$, and $\nabla_{\perp}$; and $S^{ij}(k_m, \kappa_m^2)$ are the algebraic functions shown in Eq. (50); the indices i and j run on the terms corresponding to the vectors $\bar{a}_3$, $\bar{b}$, and $\bar{b} \times \bar{a}_3$.

In the circular cylindrical coordinate system with open boundary in r and θ, the solution for Eq. (47) is:

$$
g_{mn}(\bar{r}_\perp / \bar{r}_{0\perp}) = \frac{(\text{Constant})}{\nu^2} e^{jn(\theta-\theta_0)} \begin{cases} J_n(\kappa_m r_0)\, H_n^{(2)}(\kappa_m r), & r \geq r_0 ; \\ H_n^{(2)}(\kappa_m r_0)\, J_n(\kappa_m r), & r \leq r_0 . \end{cases} \tag{52}
$$

Identifying $f_m(\xi_3)$ with $\cos k_m z$ and $g_m(\xi_3)$ with $\sin k_m z$, in line with Eqs. (16) to (18), the Green's dyadic valid for the case of two parallel conducting plates is (for $r \geq r_0$ only)

$$
\begin{aligned}
\bar{\bar{G}}(\bar{r}|\bar{r}_0) = {} & \frac{j}{4}\sum_{m,n} \frac{\kappa_m^2 - k_\parallel + k_\parallel \frac{k_m^2}{k_\perp}}{\tilde{\eta}_m \kappa_m^2 d} \left\{ (\nabla_\perp H_n^{(2)} \times \bar{a}_3)(\nabla_\perp^0 J_n \times \bar{a}_3) \cos k_m z \cos k_m z_0 \right\} \\
& + \frac{j}{4}\sum_{m,n} \frac{(\kappa_m^2 - k_\parallel)(T_m^2 - k_\perp)}{\tilde{\eta}_m k_\perp \kappa_m^2 d} (\nabla_\perp H_n^{(2)})(\nabla_\perp^0 J_n) \cos k_m z \cos k_m z_0 \\
& + \frac{j}{4}\sum_{m,n} \frac{k_T^2 - (k_\perp - k_m^2)(T_m^2 - k_\perp)}{\tilde{\eta}_m k_\perp d} H_n^{(2)} J_n \sin k_m z \sin k_m z_0\, \bar{a}_3 \bar{a}_3 \\
& - \frac{j}{4}\sum_{m,n} \frac{k_T(\kappa_m^2 - k_\parallel)}{\tilde{\eta}_m k_\perp \kappa_m^2 d} \left\{ (\nabla_\perp H_n^{(2)} \times \bar{a}_3)(\nabla_\perp^0 J_n) \cos k_m z \cos k_m z_0 \right. \\
& \qquad \left. + (\nabla_\perp H_n^{(2)})(\nabla_\perp^0 J_n \times \bar{a}_3) \cos k_m z \cos k_m z_0 \right\} \\
& + \frac{j}{4}\sum_{m,n} \frac{k_T k_m}{\tilde{\eta}_m k_\perp d} \left\{ (\nabla_\perp H_n^{(2)} \times \bar{a}_3)(J_n \bar{a}_3) \cos k_m z \sin k_m z_0 \right. \\
& \qquad \left. + (H_n^{(2)} \bar{a}_3)(\nabla_\perp^0 J_n \times \bar{a}_3) \sin k_m z \cos k_m z_0 \right\} \\
& - \frac{j}{4}\sum_{m,n} \frac{k_m(T_m^2 - k_\perp)}{\tilde{\eta}_m k_\perp d} \left\{ (\nabla_\perp H_n^{(2)})(J_n \bar{a}_3) \cos k_m z \sin k_m z_0 \right. \\
& \qquad \left. + (H_n^{(2)} \bar{a}_3)(\nabla_\perp^0 J_n) \sin k_m z \cos k_m z_0 \right\}.
\end{aligned} \tag{53}
$$

THE INFINITE DOMAIN GREEN'S DYADIC

The transition of finite or semifinite domain Green's dyadic to the infinite domain Green's dyadic may be obtained through a limiting process. Since the infinite domain Green's function has been derived by a number of authors and employed extensively,[1,2,3] no attempt will be made here to derive the infinite domain Green's dyadic into its final form. The main purpose here is to show that such transition is possible.

As an example, take the Green's dyadic of the two parallel plates case given in Eq. (53). As the plates recede to infinity, i.e. $d \to \infty$, the summation on m goes over to an integral. Written in the symbolic form of Eq. (51), the transformed infinite domain Green's dyadic, $\bar{\bar{g}}(\bar{r}|\bar{r}_0)$, is:

$$\bar{\bar{g}}(\bar{r}|\bar{r}_0) = \sum_{i,j} \sum_{n} (\bar{\mathcal{F}}^{i})(\bar{\mathcal{F}}_0^{j}) \int_0^{\infty} S^{i,j}(k)\, t_n f f^* \, dk \,. \tag{54}$$

In Eq. (54) the order of integration and differentiation operations has been interchanged and the subscript m has been dropped. Evidently, the Green's dyadic for infinite domain can be obtained through a set of auxiliary scalar functions, $I^{i,j}$, as represented by the integral in Eq. (54). In view of Eq. (52), Eq. (44), and the fact that ff^* is an even function, after using the addition theorem to perform the summation on n, $I^{i,j}$ may be written

$$I^{i,j} = \frac{1}{2} \int_{-\infty}^{\infty} S^{i,j} H_0^{(2)}(\kappa r_\perp') e^{-jkz'} \, dk \,, \tag{55}$$

where

$$r_\perp' = |\bar{r}_\perp - \bar{r}_{0\perp}| \quad , \quad z' = |z - z_0| \,.$$

It should be noted that the new coordinate system has its origin at the source point. This choice of a new origin may require subsequent transformation back to the original origin.

An exact solution of Eq. (55) is tedious and is not easily attainable; however, an asymptotic solution which is valid for waves at large distance from the source may be obtained through the method of steepest descent.

Assuming that interest is in the accuracy of the solution only to the order of $\frac{1}{r_\perp'}$, the zero order Hankel function may be expanded into its asymptotic form. Retaining only the first term, Eq. (55) then becomes:

$$I^{i,j} = \frac{e^{-j\frac{\pi}{4}}}{2}\int_{-\infty}^{\infty} \frac{S^{i,j}\, e^{-jkz'} e^{-jKr_\perp'}}{(\pi K r_\perp')^{1/2}}\, dk . \tag{56}$$

At this point, it would deem more convenient to change the coordinate system from that of circular cylindrical coordinates to that of spherical coordinates (R, φ, α); where:

$$r_\perp' = R \sin\varphi ,$$
$$z' = R \cos\varphi .$$

Under the new coordinate system, Eq. (56) becomes:

$$I^{i,j}(\bar{R}) = \frac{e^{-j\frac{\pi}{4}}}{2}\int_{-\infty}^{\infty} \frac{S^{i,j}}{(\pi K R\sin\varphi)^{1/2}}\, e^{-jR(K\sin\varphi + k\cos\varphi)}\, dk . \tag{57}$$

It is recalled that:

$$T^2 = K^2 + k^2 . \tag{58}$$

T is therefore the total propagation factor. Now for the sake of convenience, instead of k, a new integration parameter, γ, may be employed, such that

$$\left.\begin{aligned} K &= T\sin\gamma , \\ k &= T\cos\gamma . \end{aligned}\right\} \tag{59}$$

The parameter γ has the same significance as the angle which measures the wave normal if T is a constant; however, in the present case T is not a constant. In fact, combining Eq. (12), Eq. (58) and Eq. (59) yields an expression for T in terms of τ,

$$T_{1,2}^2 = \frac{-[k_\perp(k_\perp - k_\parallel) - k_T^2]\sin^2\tau + 2k_\perp k_\parallel \pm \{[k_\parallel(k_\perp - k_\parallel) - k_T^2]^2 \sin^4\tau + 4k_\parallel^2 k_T^2 \cos^2\tau\}^{1/2}}{2[k_\perp \sin^2\tau + k_\parallel \cos^2\tau]}, \tag{60}$$

where, the subscript 1 and 2 on T^2 represents the choice of plus or minus sign in Eq. (60). For simplicity, the subscripts on T^2 are dropped, assuming that it is permissible to work with one wave at a time. The integral for $I^{\lambda,j}$ becomes

$$I^{\lambda,j} = \frac{1}{(j4R\sin\varphi)^{1/2}} \int_C \frac{S^{\lambda,j}}{[\pi T(\tau)\sin\tau]^{1/2}} e^{-RU(\tau)} d\tau, \tag{61}$$

where

$$U(\tau) = j T(\tau) \cos(\tau - \varphi). \tag{62}$$

Examination of the exponent shows that the real part of $U(\tau)$ approaches $+\infty$ as k approaches $\pm\infty$. The saddle point of the integration is determined by:

$$\frac{dU}{d\tau} = 0, \tag{63}$$

which yields:

$$\frac{1}{T(\tau_0)}\left(\frac{d}{d\tau}T(\tau)\right)_{\tau=\tau_0} = \tan(\tau_0 - \varphi). \tag{64}$$

The contour of integration, C, is then chosen such that the path goes through the saddle point, τ_0, and that the imaginary part of U is constant. Following the method of steepest descent, the solution for Eq. (61) is therefore

$$I^{\lambda,j} = \frac{S^{\lambda,j}_{(\tau_0)}}{2n(\tau_0)} \frac{e^{-RU(\tau_0)}}{R(\sin\varphi)^{1/2}}, \tag{65}$$

where

$$\eta(\tau_0) = \left\{2T\sin\tau\left[\left(\frac{dT}{d\tau} - T\right)\cos(\tau-\varphi) - 2\frac{dT}{d\tau}\sin(\tau-\varphi)\right]\right\}^{1/2}_{\tau=\tau_0} . \tag{66}$$

The electric field intensity $\bar{E}$ in infinite domain may, therefore, be obtained from:

$$\bar{E} = \sum_{i,j}\int_{V_0}\left[(\bar{\mathcal{Z}}^{i})(\bar{\mathcal{Z}}_0^{j})\,\frac{S^{i,j}}{2\eta(\tau_0)}\,\frac{e^{-RU(\tau_0)}}{(R^2\sin\varphi)^{1/2}}\right]\cdot\bar{J}_s\,dV_0\,, \tag{67}$$

providing that all parameters, including the differential operators are properly transformed to the correct observer and source coordinates in the spherical coordinate system.

CONCLUSION

The results of separability studies in this work indicate that the dyadic-vector Helmholtz equation is solvable by the separation technique in four cylindrical coordinate systems. It may be noted that the solutions obtained by the separation technique are uncoupled, i.e., it is possible to solve for one field of the waves without explicit knowledge of the other field. Such simplicity may be contrasted to the coupled field solution that often prevailed in the past. In the past, free wave solutions in a bounded anisotropic plasma often have been obtained by direct manipulations of Maxwell's equations and the generalized Ohm's law. Such manipulations often led to second order differential equations such that the fields are coupled, i.e. the electric field is solvable in terms of the magnetic field and vice versa. Except for some special cases, to uncouple the fields, the order of the differential equations must be raised beyond two and thereby increases the difficulty in obtaining solutions in simple form.

The Green's dyadic constructed through sets of eigenfunctions for finite or semifinite domain problems is expressed in terms of differential operators which have the advantage of ease of operation over integral operators.

The form of solutions for infinite domain problems as shown in Eq.(67) is not exactly of the same form obtained by Bunkin.[1] The most noticeable difference lies in the manner of operation. Bunkin's solution requires two second order differential operations, while Eq. (67) requires only two first order differential operations. However, the result of Eq.(67) compares favorably with that obtained by Bunkin.

APPENDIX A

The inhomogeneous equation is

$$\nabla\times\nabla\times\bar{E} - \bar{\bar{k}}\cdot\bar{E} = \bar{J}_s \,. \tag{A-1}$$

A Green's equation is assumed

$$\nabla\times\nabla\times\bar{\bar{G}} - \bar{\bar{K}}\cdot\bar{\bar{G}} = \bar{\bar{I}}\,\delta(\bar{r}-\bar{r}_0)\,. \tag{A-2}$$

Multiply $\bar{\bar{G}}$ from the right into Eq. (A-1) and multiply $\bar{E}$ from the left into Eq. (A-2), subtract and integrate over the entire space on the source coordinate yielding

$$\bar{E}(\bar{r}) = \int \bar{\bar{G}}\cdot\bar{J}_s\,dv_0 + \int\left\{\bar{E}\cdot\nabla\times\nabla\times\bar{\bar{G}} - (\nabla\times\nabla\times\bar{E})\cdot\bar{\bar{G}}\right\}dv_0 - \int\left\{\bar{E}\cdot\bar{\bar{K}}\cdot\bar{\bar{G}} - \bar{\bar{k}}\cdot\bar{E}\cdot\bar{\bar{G}}\right\}dv_0\,. \tag{A-3}$$

Using Green's theorem the second integral can be transformed into a surface integral. If the Green's dyadic satisfies the same boundary conditions the

$\overline{E}$ field statisfies, the surface integral vanishes.

The dyadics $\overline{\overline{k}}$ and $\overline{\overline{K}}$ in the third integral may be expressed in terms of their symmetrical components (subscript s) and antisymmetrical components (subscript a)

$$\overline{\overline{k}} = \overline{\overline{k}}_s + \overline{\overline{k}}_a, \tag{A-4}$$

$$\overline{\overline{K}} = \overline{\overline{K}}_s + \overline{\overline{K}}_a. \tag{A-5}$$

Substituting into the integral, assuming $\overline{\overline{G}}$ being symmetric and reciprocal with respect to $\overline{r}$ and $\overline{r}_o$, it is found that

$$\overline{E}\cdot\overline{\overline{K}}_s\cdot\overline{\overline{G}} - \overline{\overline{k}}_s\cdot\overline{E}\cdot\overline{\overline{G}} = 0, \quad \text{if} \quad \overline{\overline{K}}_s = \overline{\overline{k}}_s \;;$$

and

$$\overline{E}\cdot\overline{\overline{K}}_a\cdot\overline{\overline{G}} - \overline{\overline{k}}_a\cdot\overline{E}\cdot\overline{\overline{G}} = 0, \quad \text{if} \quad \overline{\overline{K}}_a = -\overline{\overline{k}}_a.$$

Thus, it is shown that for Eq. (20) to hold, the third integral must also vanish, or

$$\overline{\overline{K}} = \overline{\overline{k}} \,. \tag{A-6}$$

APPENDIX B

Eqs. (45) and (46) in essence demand that the Green's dyadic be symmetrical. They also imply a condition for the (ξ_3, ξ_3^o) functions such that the solution for Eq. (44) must be chosen to satisfy

$$\nabla_{\parallel}\nabla_{\parallel}^{o}\left(f_m f_m^{*},\; g_m g_m^{*}\right) = k_m^2\left(f_m f_m^{*},\; g_m g_m^{*}\right)\overline{a}_3\,\overline{a}_3. \tag{B-1}$$

Eqs. (45) and (46) along with Eq. (B-1) imply

$$\nabla_{\perp}^{o}\times\overline{a}_3 = -\nabla_{\perp}\times\overline{a}_3 , \tag{B-2}$$

$$\nabla_{\perp}^{o} = -\nabla_{\perp} , \tag{B-3}$$

and

$$\nabla_{\parallel}^{o} = -\nabla_{\parallel} . \tag{B-4}$$

The symbolic form of the Green's dyadic, as is given in Eq. (51), cannot be symmetric unless the source coordinates operator $\mathcal{L}_o^{\prime}$ and the observer coordinates operator $\bar{\bar{\mathcal{L}}}^{j}$ can be interchanged. Of course, Eqs. (B-2) through (B-4) are not the only possible conditions that may force the Green's dyadic to be symmetric.

The Green's dyadic is symmetric only in the coordinate system for which the Green's dyadic is constructed. Using variational technique, a given Green's dyadic may be transformed to one that is valid for a problem of different boundary configuration in a different coordinate system. But the symmetrical property of the original Green's dyadic is not necessarily retained in the transformation. This is especially true in the case of the Green's dyadic for the infinite domain.

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